| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 1a | Makes an attempt to substitute $k = 1$, $k = 2$ and $k = 4$ into $a_k = 2^k + 1, k$ Ö 1 | **M1** | 1.1b | 5th<br>Understand disproof by counter example. |
| | Shows that $a_1 = 3$, $a_2 = 5$ and $a_4 = 17$ and these are prime numbers. | **A1** | 1.1b | |
| | | **(2)** | | |
| 1b | Substitutes a value of $k$ that does not yield a prime number.<br>For example, $a_3 = 9$ or $a_5 = 33$ | **A1** | 1.1b | 5th<br>Understand disproof by counter example. |
| | Concludes that their number is not prime.<br>For example, states that 9 = 3 × 3, so 9 is not prime. | **B1** | 2.4 | |
| | | **(2)** | | |
| | | | | **(4 marks)** |

**Notes**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| **2** | Makes an attempt to substitute any of $n$ = 1, 2, 3, 4, 5 or 6 into $\frac{n(n+1)}{2}$ | **M1** | 1.1b | 5th<br>Complete proofs by exhaustion. |
| | Successfully substitutes $n$ = 1, 2, 3, 4, 5 **and** 6 into $\frac{n(n+1)}{2}$<br>$1=\frac{(1)(2)}{2}$<br>$1+2=\frac{(2)(3)}{2}$<br>$1+2+3=\frac{(3)(4)}{2}$<br>$1+2+3+4=\frac{(4)(5)}{2}$<br>$1+2+3+4+5=\frac{(5)(6)}{2}$<br>$1+2+3+4+5+6=\frac{(6)(7)}{2}$ | **A1** | 1.1b | |
| | Draws the conclusion that as the statement is true for all numbers from 1 to 6 inclusive, it has been proved by exhaustion. | **B1** | 2.4 | |
| | | | | **(3 marks)** |

**Notes**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 3 | Begins the proof by assuming the opposite is true.<br>'Assumption: there exists a product of two odd numbers that is even.' | **B1** | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Defines two odd numbers. Can choose any two different variables.<br>'Let $2m + 1$ and $2n + 1$ be our two odd numbers.' | **B1** | 2.2a | |
| | Successfully multiplies the two odd numbers together:<br>$(2m+1)(2n+1) \equiv 4mn+2m+2n+1$ | **M1** | 1.1b | |
| | Factors the expression and concludes that this number must be odd.<br>$4mn+2m+2n+1 \equiv 2(2mn+m+n)+1$<br>$2(2mn+m+n)$ is even, so $2(2mn+m+n)+1$ must be odd. | **M1** | 1.1b | |
| | Makes a valid conclusion.<br>This contradicts the assumption that the product of two odd numbers is even, therefore the product of two odd numbers is odd. | **B1** | 2.4 | |
| | | | | **(5 marks)** |

**Notes**

**Alternative method**

Assume the opposite is true: there exists a product of two odd numbers that is even. **(B1)**

If the product is even then 2 is a factor. **(B1)**

So 2 is a factor of at least one of the two numbers. **(M1)**

So at least one of the two numbers is even. **(M1)**

This contradicts the statement that both numbers are odd. **(B1)**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 4 | Begins the proof by assuming the opposite is true.<br>'Assumption: there exists a number $n$ such that $n$ is odd and $n^3 + 1$ is also odd.' | **B1** | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Defines an odd number.<br>'Let $2k + 1$ be an odd number.' | **B1** | 2.2a | |
| | Successfully calculates $(2k+1)^3+1$<br>$(2k+1)^3+1\equiv\left(8k^3+12k^2+6k+1\right)+1\equiv 8k^3+12k^2+6k+2$ | **M1** | 1.1b | |
| | Factors the expression and concludes that this number must be even.<br>$8k^3+12k^2+6k+2\equiv 2\left(4k^3+6k^2+3k+1\right)$<br>$2\left(4k^3+6k^2+3k+1\right)$ is even. | **M1** | 1.1b | |
| | Makes a valid conclusion.<br>This contradicts the assumption that there exists a number $n$ such that $n$ is odd and $n^3 + 1$ is also odd, so if $n$ is odd, then $n^3 + 1$ is even. | **B1** | 2.4 | |
| | | | | **(5 marks)** |

**Notes**

**Alternative method**

Assume the opposite is true: there exists a number $n$ such that $n$ is odd and $n^3 + 1$ is also odd. (**B1**)

If $n^3 + 1$ is odd, then $n^3$ is even. (**B1**)

So 2 is a factor of $n^3$. (**M1**)

This implies 2 is a factor of $n$. (**M1**)

This contradicts the statement $n$ is odd. (**B1**)

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 5 | Begins the proof by assuming the opposite is true.<br>'Assumption: there do exist integers $a$ and $b$ such that $25a+15b=1$' | **B1** | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Understands that $25a+15b=1 \Rightarrow 5a+3b=\frac{1}{5}$<br>'As both 25 and 15 are multiples of 5, divide both sides by 5 to leave $5a+3b=\frac{1}{5}$' | **M1** | 2.2a | |
| | Understands that if $a$ and $b$ are integers, then $5a$ is an integer, $3b$ is an integer and $5a + 3b$ is also an integer. | **M1** | 1.1b | |
| | Recognises that this contradicts the statement that $5a+3b=\frac{1}{5}$, as $\frac{1}{5}$ is not an integer. Therefore there do not exist integers $a$ and $b$ such that $25a+15b=1$' | **B1** | 2.4 | |
| | | | | **(4 marks)** |

**Notes**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 6 | Begins the proof by assuming the opposite is true.<br>'Assumption: there exists a rational number $\frac{a}{b}$ such that $\frac{a}{b}$ is the greatest positive rational number.' | B1 | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Makes an attempt to consider a number that is clearly greater than $\frac{a}{b}$:<br>'Consider the number $\frac{a}{b}+1$, which must be greater than $\frac{a}{b}$' | M1 | 2.2a | |
| | Simplifies $\frac{a}{b}+1$ and concludes that this is a rational number.<br>$\frac{a}{b}+1 \equiv \frac{a}{b}+\frac{b}{b} \equiv \frac{a+b}{b}$<br>By definition, $\frac{a+b}{b}$ is a rational number. | M1 | 1.1b | |
| | Makes a valid conclusion.<br>This contradicts the assumption that there exists a greatest positive rational number, so we can conclude that there is not a greatest positive rational number. | B1 | 2.4 | |
| | | | | **(4 marks)** |

**Notes**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 7 | Begins the proof by assuming the opposite is true.<br>'Assumption: given a rational number $a$ and an irrational number $b$, assume that $a - b$ is rational.' | **B1** | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Sets up the proof by defining the different rational and irrational numbers. The choice of variables does not matter.<br>Let $a = \frac{m}{n}$<br>As we are assuming $a - b$ is rational, let $a - b = \frac{p}{q}$<br>So $a - b = \frac{p}{q} \Rightarrow \frac{m}{n} - b = \frac{p}{q}$ | **M1** | 2.2a | |
| | Solves $\frac{m}{n} - b = \frac{p}{q}$ to make $b$ the subject and rewrites the resulting expression as a single fraction:<br>$\frac{m}{n} - b = \frac{p}{q} \Rightarrow b = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq}$ | **M1** | 1.1b | |
| | Makes a valid conclusion.<br>$b = \frac{mq - pn}{nq}$, which is rational, contradicts the assumption $b$ is an irrational number. Therefore the difference of a rational number and an irrational number is irrational. | **B1** | 2.4 | |
| | | | | **(4 marks)** |

**Notes**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 8 | Begins the proof by assuming the opposite is true.<br>'Assumption: there exist positive integer solutions to the statement $x^2 - y^2 = 1$' | **B1** | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Sets up the proof by factorising $x^2 - y^2$ and stating $(x-y)(x+y) = 1$ | **M1** | 2.2a | |
| | States that there is only one way to multiply to make 1:<br>$1 \times 1 = 1$<br>and concludes this means that:<br>$x - y = 1$<br>$x + y = 1$ | **M1** | 1.1b | |
| | Solves this pair of simultaneous equations to find the values of $x$ and $y$: $x = 1$ and $y = 0$ | **M1** | 1.1b | |
| | Makes a valid conclusion.<br>$x = 1$, $y = 0$ are not both positive integers, which is a contradiction to the opening statement. Therefore there do not exist positive integers $x$ and $y$ such that $x^2 - y^2 = 1$ | **B1** | 2.4 | |
| | | | | **(5 marks)** |

**Notes**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| **9a** | Begins the proof by assuming the opposite is true.<br>‘Assumption: there exists a number $n$ such that $n^2$ is even and $n$ is odd.’ | **B1** | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Defines an odd number (choice of variable is not important) and successfully calculates $n^2$<br>Let $2k + 1$ be an odd number.<br>$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ | **M1** | 2.2a | |
| | Factors the expression and concludes that this number must be odd.<br>$4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1$, so $n^2$ is odd. | **M1** | 1.1b | |
| | Makes a valid conclusion.<br>This contradicts the assumption $n^2$ is even. Therefore if $n^2$ is even, $n$ must be even. | **B1** | 2.4 | |
| | | **(4)** | | |

| | | | | |
|---|---|---|---|---|
| **9b** | Begins the proof by assuming the opposite is true.<br>'Assumption: $\sqrt{2}$ is a rational number.' | **B1** | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Defines the rational number:<br>$\sqrt{2}=\frac{a}{b}$ for some integers $a$ and $b$, where $a$ and $b$ have no common factors. | **M1** | 2.2a | |
| | Squares both sides and concludes that $a$ is even:<br>$\sqrt{2}=\frac{a}{b}\Rightarrow 2=\frac{a^2}{b^2}\Rightarrow a^2=2b^2$<br>From part **a**: $a^2$ is even implies that $a$ is even. | **M1** | 1.1b | |
| | Further states that if $a$ is even, then $a = 2c$. Choice of variable is not important. | **M1** | 1.1b | |
| | Makes a substitution and works through to find $b^2=2c^2$, concluding that $b$ is also even.<br>$a^2=2b^2\Rightarrow(2c)^2=2b^2\Rightarrow 4c^2=2b^2\Rightarrow b^2=2c^2$<br>From part **a**: $b^2$ is even implies that $b$ is even. | **M1** | 1.1b | |
| | Makes a valid conclusion.<br>If $a$ and $b$ are even, then they have a common factor of 2, which contradicts the statement that $a$ and $b$ have no common factors. Therefore $\sqrt{2}$ is an irrational number. | **B1** | 2.4 | |
| | | **(6)** | | |
| | | | | **(10 marks)** |

**Notes**

| Q | Scheme | Marks | AOs | Pearson Progression Step and Progress descriptor |
|---|---|---|---|---|
| 10 | Begins the proof by assuming the opposite is true.<br>'Assumption: there is a finite amount of prime numbers.' | B1 | 3.1 | 7th<br>Complete proofs using proof by contradiction. |
| | Considers what having a finite amount of prime numbers means by making an attempt to list them:<br>Let all the prime numbers exist be $p_1, p_2, p_3, ...p_n$ | M1 | 2.2a | |
| | Consider a new number that is one greater than the product of all the existing prime numbers:<br>Let $N = (p_1 \times p_2 \times p_3 \times ... \times p_n) + 1$ | M1 | 1.1b | |
| | Understands the implication of this new number is that division by any of the existing prime numbers will leave a remainder of 1. So none of the existing prime numbers is a factor of *N*. | M1 | 1.1b | |
| | Concludes that either *N* is prime or *N* has a prime factor that is not currently listed. | B1 | 2.4 | |
| | Recognises that either way this leads to a contradiction, and therefore there is an infinite number of prime numbers. | B1 | 2.4 | |
| | | | | **(6 marks)** |

**Notes**

If *N* is prime, it is a new prime number separate to the finite list of prime numbers, $p_1, p_2, p_3, ...p_n$.

If *N* is divisible by a previously unknown prime number, that prime number is also separate to the finite list of prime numbers.